Structural Analysis III

The Moment Area Method – Mohr's Theorems

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1. Introduction

1.1 Purpose

The moment-area method, developed by Mohr, is a powerful tool for finding the deflections of structures primarily subjected to bending. Its ease of finding deflections of determinate structures makes it ideal for solving indeterminate structures, using compatibility of displacement.

We will examine compatibility of displacement in more detail later, but its essence is the knowledge of certain displacements. For example, we know that the displacement of a simply supported beam is zero at each support. We will use this information, in association with Mohr's Theorems, to solve for related indeterminate beams.

2. Theory

2.1 Basis

We consider a length of beam *AB* in its undeformed and deformed state, as shown on the next page. Studying this diagram *carefully*, we note:

1. *AB* is the original unloaded length of the beam and *A'B'* is the deflected position of *AB* when loaded.

2. The angle subtended at the centre of the arc *A'OB'* is θ and is the change in curvature from *A'* to *B'*.

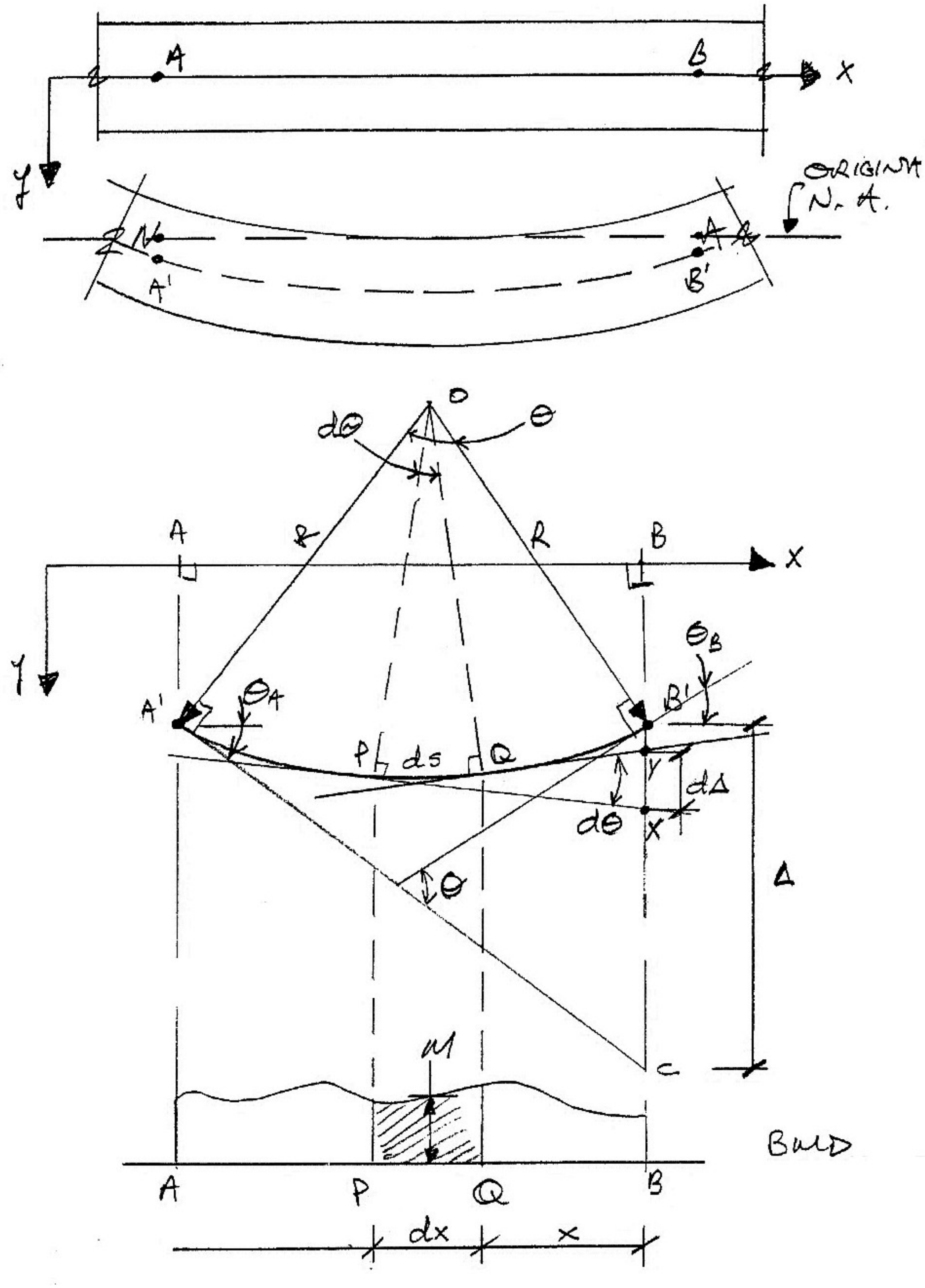
3. *PQ* is a very short length of the beam, measured as ds along the curve and dx along the *x*-axis.

4. $d\theta$ is the angle subtended at the centre of the arc ds.

5. $d\theta$ is the change in curvature from *P* to *Q*.

6. *M* is the average bending moment over the portion dx between *P* and *Q*.

7. The distance Δ is known as the vertical intercept and is the distance from *B'* to the produced tangent to the curve at *A'* which crosses under *B'* at *C*. It is measured perpendicular to the undeformed neutral axis (i.e. the *x*-axis) and so is 'vertical'.



Basis of Theory

2.2 Mohr's First Theorem (Mohr I)

Development

Noting that the angles are always measured in radians, we have:

$$ds = R \cdot d\theta$$
$$\therefore R = \frac{ds}{d\theta}$$

From the Euler-Bernoulli Theory of Bending, we know:

$$\frac{1}{R} = \frac{M}{EI}$$

Hence:

$$d\theta = \frac{M}{EI} \cdot ds$$

But for small deflections, the chord and arc length are similar, i.e. $ds \approx dx$, giving:

$$d\theta = \frac{M}{EI} \cdot dx$$

The total change in rotation between *A* and *B* is thus:

$$\int_A^B d\theta = \int_A^B \frac{M}{EI}\, dx$$

The term M/EI is the curvature and the diagram of this terms as it changes along a beam is the curvature diagram (or more simply the M/EI diagram). Thus we have:

$$d\theta_{BA} = \theta_B - \theta_A = \int_A^B \frac{M}{EI}\,dx$$

This is interpreted as:

$$\boxed{\left[\text{Change in slope}\right]_{AB} = \left[\text{Area of } \frac{M}{EI} \text{ diagram}\right]_{AB}}$$

This is Mohr's First Theorem (Mohr I):

> *The change in slope over any length of a member subjected to bending is equal to the area of the curvature diagram over that length.*

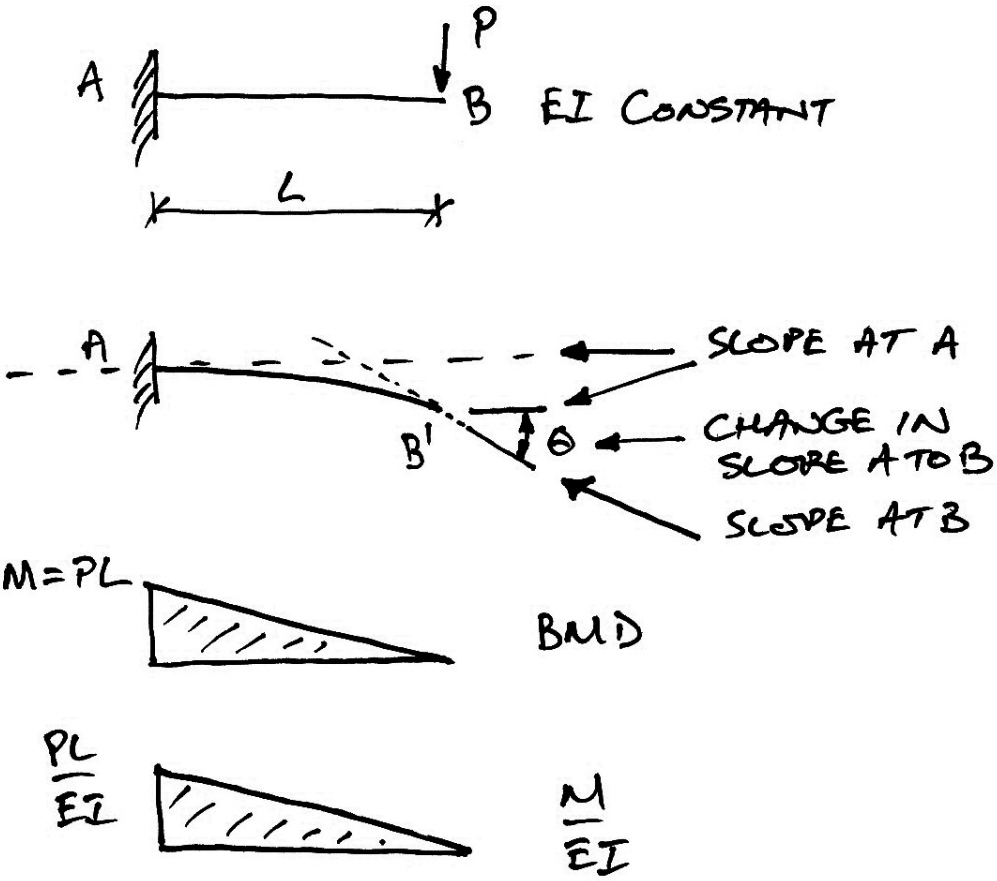
Usually the beam is prismatic and so *E* and *I* do not change over the length *AB*, whereas the bending moment *M* will change. Thus:

$$\theta_{AB} = \frac{1}{EI}\int_A^B M\;dx$$

$$\left[\text{Change in slope}\right]_{AB} = \frac{\left[\text{Area of } M \text{ diagram}\right]_{AB}}{EI}$$

Example 1

For the cantilever beam shown, we can find the rotation at *B* easily:



Thus, from Mohr I, we have:

$$\left[\text{Change in slope}\right]_{AB} = \left[\text{Area of } \frac{M}{EI} \text{ diagram}\right]_{AB}$$

$$\theta_B - \theta_A = \frac{1}{2} \cdot L \cdot \frac{PL}{EI}$$

Since the rotation at *A* is zero (it is a fixed support), i.e. $\theta_A = 0$, we have:

$$\theta_B = \frac{PL^2}{2EI}$$

2.3 Mohr's Second Theorem (Mohr II)

Development

From the main diagram, we can see that:

$$d\Delta = x \cdot d\theta$$

But, as we know from previous,

$$d\theta = \frac{M}{EI} \cdot dx$$

Thus:

$$d\Delta = \frac{M}{EI} \cdot x \cdot dx$$

And so for the portion *AB*, we have:

$$\begin{aligned} \int_A^B d\Delta &= \int_A^B \frac{M}{EI} \cdot x \cdot dx \\ \Delta_{BA} &= \left[\int_A^B \frac{M}{EI} \cdot dx \right] \bar{x} \\ &= \text{ First moment of } \frac{M}{EI} \text{ diagram about } B \end{aligned}$$

This is easily interpreted as:

$$\begin{bmatrix}\text{Vertical}\\\text{Intercept}\end{bmatrix}_{BA} = \begin{bmatrix}\text{Area of}\\\dfrac{M}{EI}\text{ diagram}\end{bmatrix}_{BA} \times \begin{bmatrix}\text{Distance from } B \text{ to centroid}\\\text{of } \left(\dfrac{M}{EI}\right)_{BA} \text{ diagram}\end{bmatrix}$$

This is Mohr's Second Theorem (Mohr II):

For an originally straight beam, subject to bending moment, the vertical intercept between one terminal and the tangent to the curve of another terminal is the first moment of the curvature diagram about the terminal where the intercept is measured.

There are two crucial things to note from this definition:

- Vertical intercept is *not* deflection; look again at the fundamental diagram – it is the distance from the deformed position of the beam to the tangent of the deformed shape of the beam at another location. That is:

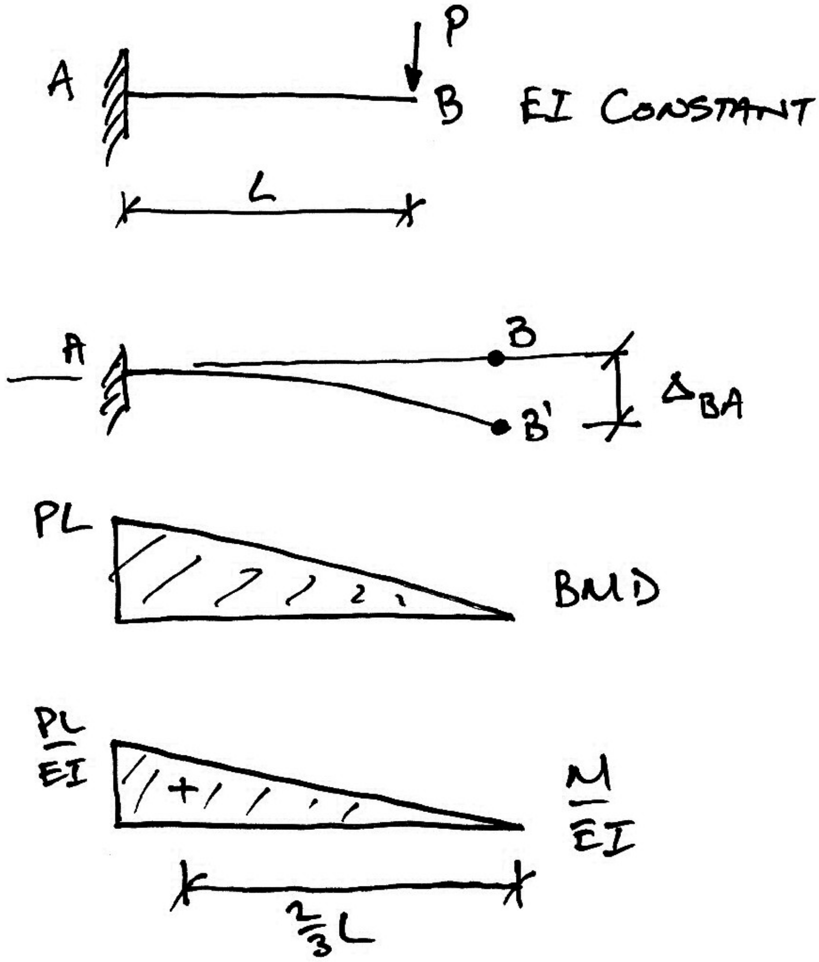
$$\Delta \neq \delta$$

- The moment of the curvature diagram must be taken about the point where the vertical intercept is required. That is:

$$\Delta_{BA} \neq \Delta_{AB}$$

Example 2

For the cantilever beam, we can find the defection at *B* since the produced tangent at *A* is horizontal, i.e. $\theta_A = 0$. Thus it can be used to measure deflections from:



Thus, from Mohr II, we have:

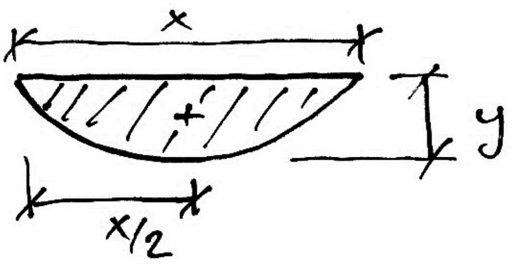
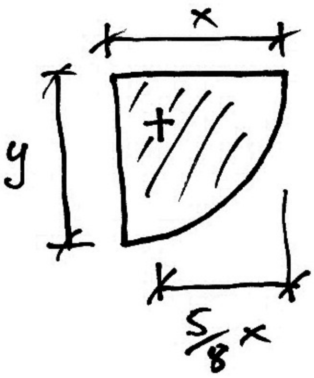
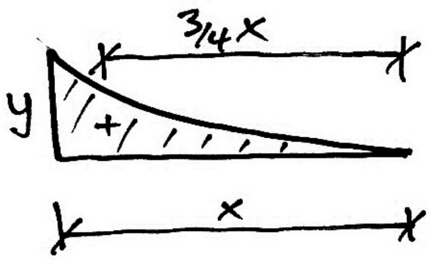
$$\Delta_{BA} = \left[\frac{1}{2} \cdot L \cdot \frac{PL}{EI}\right]\left[\frac{2L}{3}\right]$$

And so the deflection at *B* is:

$$\delta_B = \frac{PL^2}{3EI}$$

2.4 Area Properties

These are well known for triangular and rectangular areas. For parabolic areas we have:

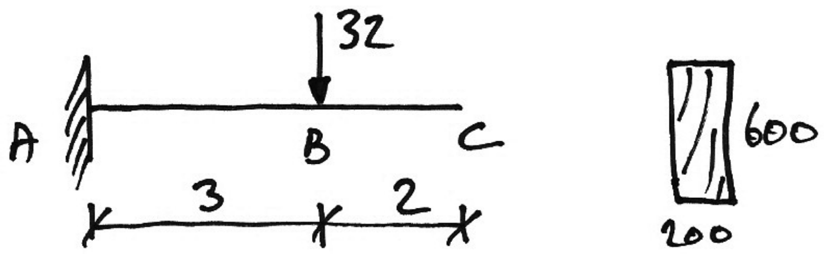
Shape	**Area**	**Centroid**
	$A = \frac{2}{3}xy$	$\bar{x} = \frac{1}{2}x$
	$A = \frac{2}{3}xy$	$\bar{x} = \frac{5}{8}x$
	$A = \frac{1}{3}xy$	$\bar{x} = \frac{3}{4}x$

3. Application to Determinate Structures

3.1 Basic Examples

Example 3

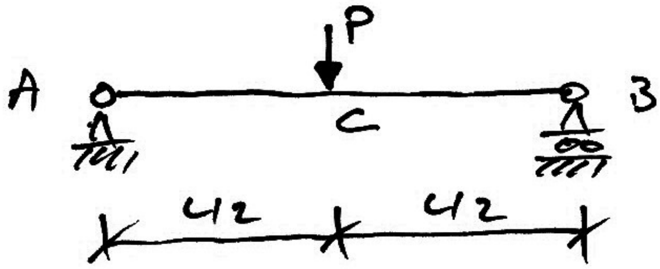
For the following beam, find δ_B, δ_C, θ_B and θ_C given the section dimensions shown and $E = 10$ kN/mm^2.

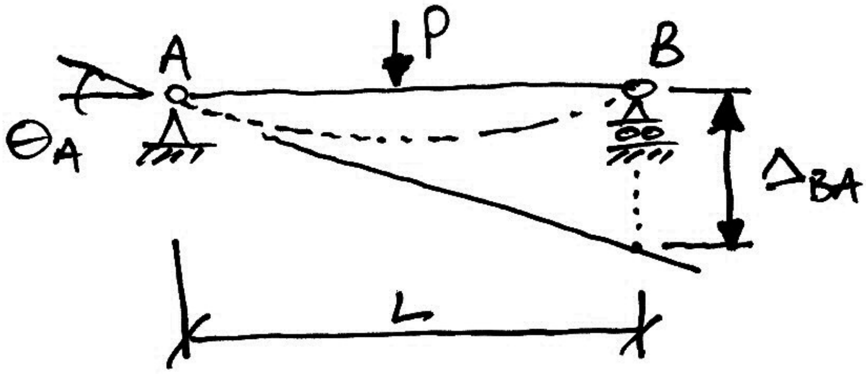


To be done in class.

Example 4

For the following simply-supported beam, we can find the rotation at A using Mohr's Second Theorem. The deflected shape diagram is used to identify relationships between vertical intercepts and rotations:



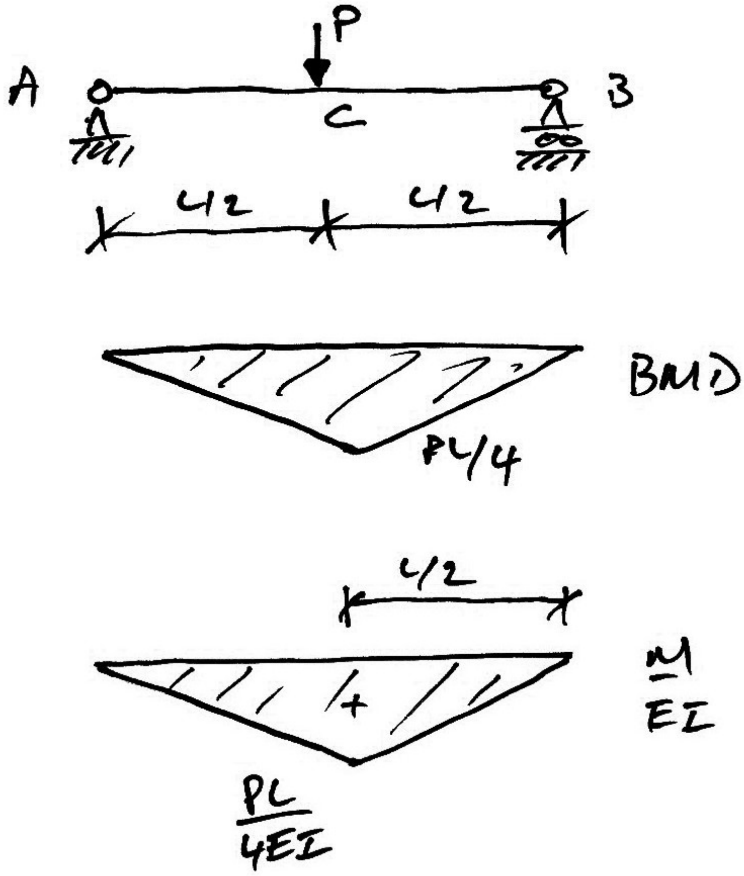


The key to the solution here is that we can calculate Δ_{BA} using Mohr II but from the diagram we can see that we can use the formula $S = R\theta$ for small angles:

$$\Delta_{BA} = L \cdot \theta_A$$

Therefore once we know Δ_{BA} using Mohr II, we can find $\theta_A = \Delta_{BA}/L$.

To calculate Δ_{BA} using Mohr II we need the bending moment and curvature diagrams:



Thus, from Mohr II, we have:

$$\begin{aligned}\Delta_{BA} &= \left[\frac{1}{2}\cdot L\cdot\frac{PL}{4EI}\right]\left[\frac{L}{2}\right]\\ &= \frac{PL^3}{16EI}\end{aligned}$$

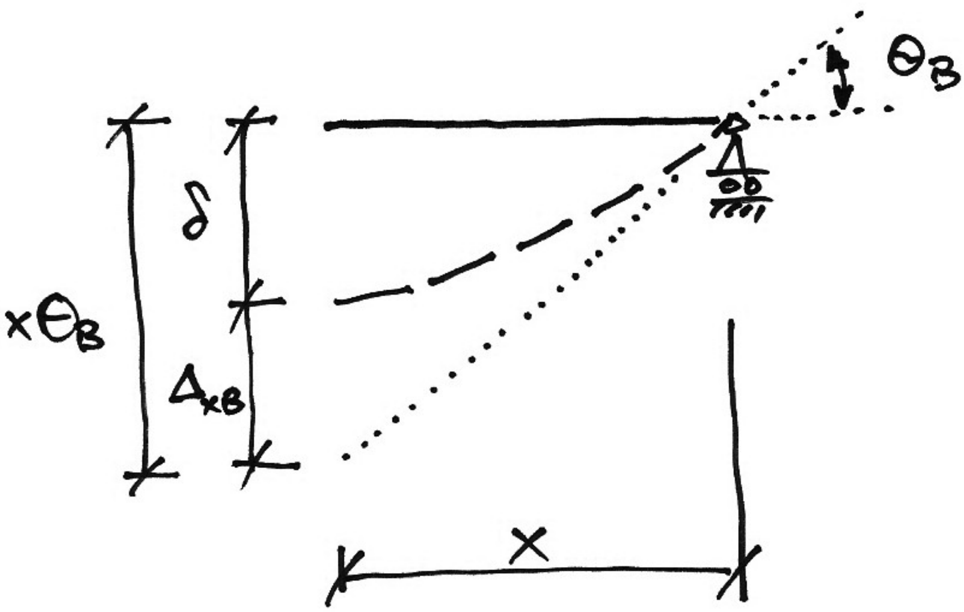
But, $\Delta_{BA} = L\cdot\theta_A$ and so we have:

$$\begin{aligned}\theta_A &= \frac{\Delta_{BA}}{L}\\ &= \frac{PL^2}{16EI}\end{aligned}$$

3.2 Finding Deflections

General Procedure

To find the deflection at any location *x* from a support use the following relationships between rotations and vertical intercepts:



Thus we:

1. Find the rotation at the support using Mohr II as before;
2. For the location *x*, and from the diagram we have:

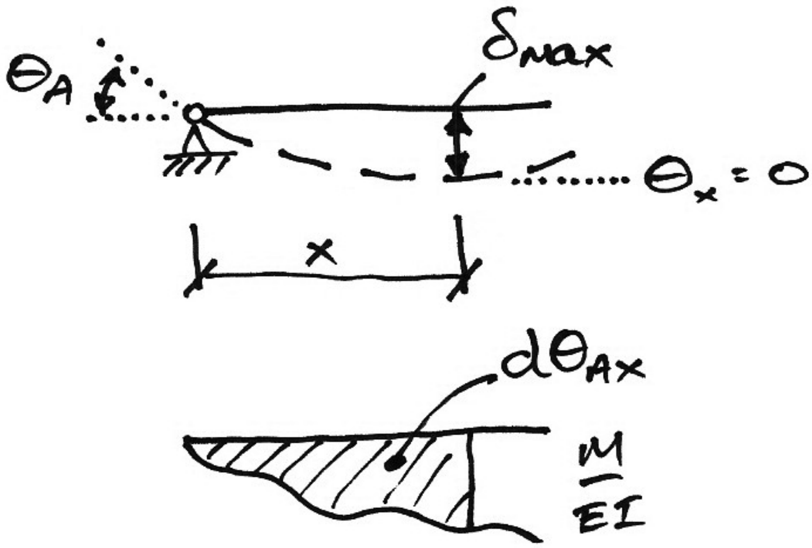
$$\delta_x = x \cdot \theta_B - \Delta_{xB}$$

Maximum Deflection

To find the maximum deflection we first need to find the location at which this occurs. We know from beam theory that:

$$\delta = \frac{d\theta}{dx}$$

Hence, from basic calculus, the maximum deflection occurs at a rotation, $\theta = 0$:



To find where the rotation is zero:

1. Calculate a rotation at some point, say support *A*, using Mohr II say;
2. Using Mohr I, determine at what distance from the point of known rotation (*A*) the change in rotation (Mohr I), $d\theta_{Ax}$ equals the known rotation (θ_A).
3. This is the point of maximum deflection since:

$$\theta_A - d\theta_{Ax} = \theta_A - \theta_A = 0$$

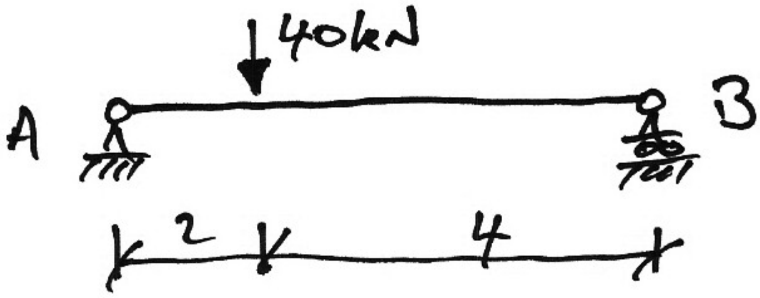
Example 5

For the following beam of constant *EI*:

(a) Determine θ_A, θ_B and δ_C;

(b) What is the maximum deflection and where is it located?

Give your answers in terms of *EI*.

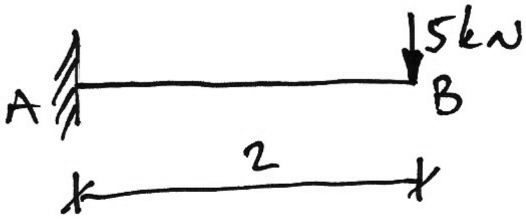


To be done in class.

3.3 Problems

1. For the beam of Example 3, using only Mohr's First Theorem, show that the rotation at support *B* is equal in magnitude but not direction to that at *A*.

2. For the following beam, of dimensions $b = 150 \text{ mm}$ and $d = 225 \text{ mm}$ and $E = 10 \text{ kN/mm}^2$, show that $\theta_B = 7 \times 10^{-4}$ rads and $\delta_B = 9.36 \text{ mm}$.



3. For a cantilever *AB* of length *L* and stiffness *EI*, subjected to a UDL, show that:

$$\theta_B = \frac{wL^3}{6EI}; \quad \delta_B = \frac{wL^4}{8EI}$$

4. For a simply-supported beam *AB* with a point load at mid span (*C*), show that:

$$\delta_C = \frac{PL^3}{48EI}$$

5. For a simply-supported beam *AB* of length *L* and stiffness *EI*, subjected to a UDL, show that:

$$\theta_A = \frac{wL^3}{24EI}; \quad \theta_B = -\frac{wL^3}{24EI}; \quad \delta_C = \frac{5wL^4}{384EI}$$

4. Application to Indeterminate Structures

4.1 Basis of Approach

Using the principle of superposition we will separate indeterminate structures into a primary and reactant structures.

For these structures we will calculate the deflections at a point for which the deflection is known in the original structure.

We will then use compatibility of displacement to equate the two calculated deflections to the known deflection in the original structure.

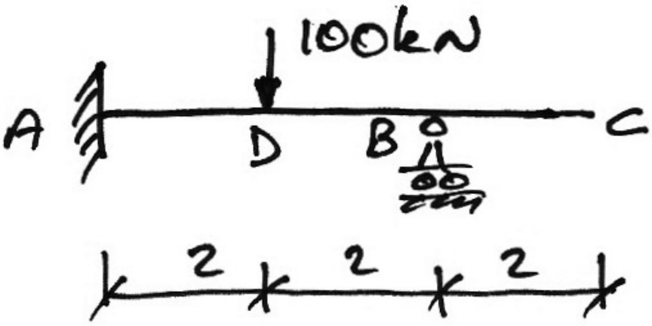
Doing so will yield the value of the redundant reaction chosen for the reactant structure.

Once this is known all other load effects (bending, shear, deflections, rotations) can be calculated.

See the handout on Compatibility of Displacement and the Principle of Superposition for more on this approach.

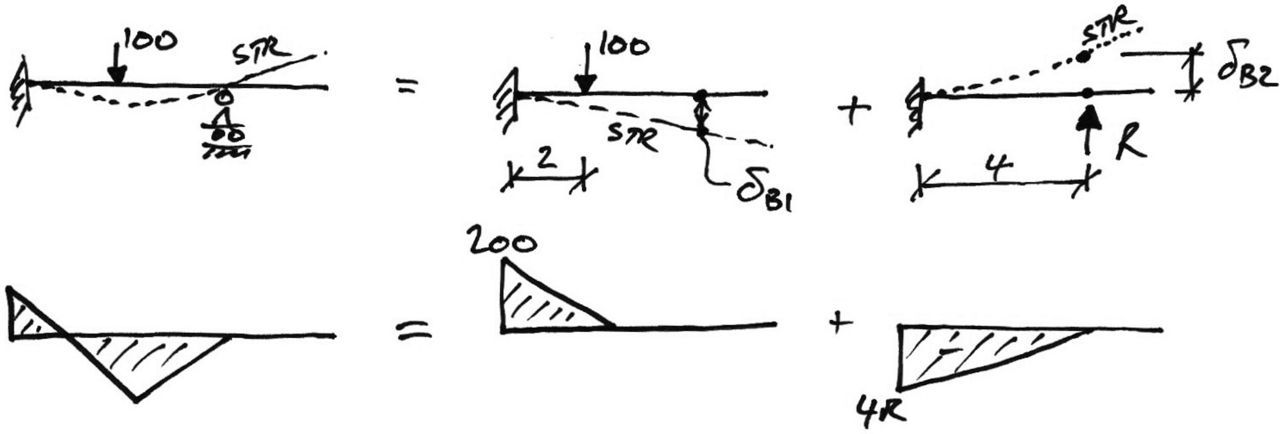
4.2 Example 6: Propped Cantilever

For the following prismatic beam, find the maximum deflection in span *AB* and the deflection at C in terms of *EI*.



Find the reaction at B

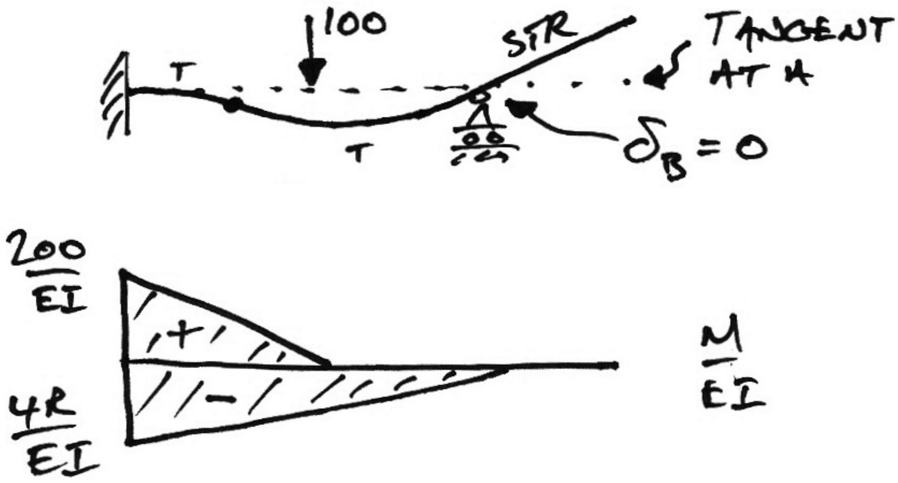
Since this is an indeterminate structure, we first need to solve for one of the unknown reactions. Choosing V_B as our redundant reaction, using the principle of superposition, we can split the structure up as shown:



(a) = **(b)** + **(c)**

In which *R* is the value of the chosen redundant.

In the final structure (a) we know that the deflection at *B*, δ_B, must be zero as it is a roller support. So from the BMD that results from the superposition of structures (b) and (c) we can calculate δ_B in terms of *R* and solve since $\delta_B = 0$.



We have from Mohr II:

$$EI\Delta_{BA} = \left[\left(\frac{1}{2}\cdot 2\cdot 200\right)\left(2+\frac{2}{3}\cdot 2\right)\right]_{(b)} + \left[-\left(\frac{1}{2}\cdot 4\cdot 4R\right)\left(\frac{2}{3}\cdot 4\right)\right]_{(c)}$$

$$= \frac{2000}{3} - \frac{64}{3}R$$

$$= \frac{1}{3}\left(2000 - 64R\right)$$

But since $\theta_A = 0$, $\delta_B = \Delta_{BA}$ and so we have:

$$EI\Delta_{BA} = 0$$

$$\frac{1}{3}\left(2000 - 64R\right) = 0$$

$$64R = 2000$$

$$R = +31.25 \text{ kN}$$

The positive sign for *R* means that the direction we originally assumed for it (upwards) was correct.

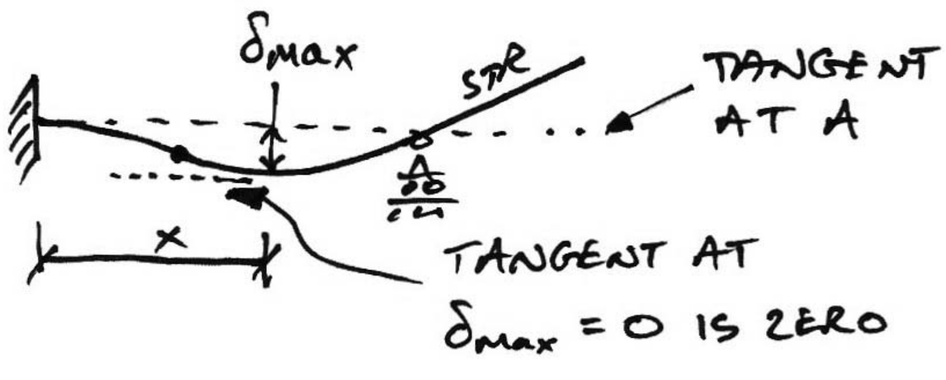
At this point the final BMD can be drawn but since its shape would be more complex we continue to operate using the structure (b) and (c) BMDs.

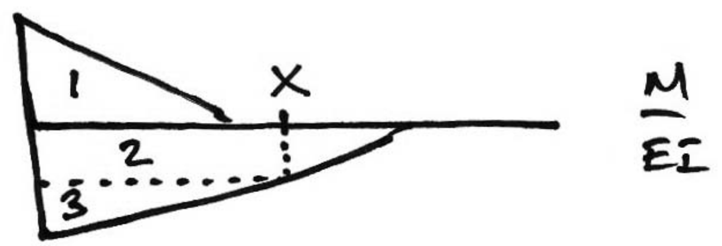
Find the location of the maximum deflection

This is the next step in determining the maximum deflection in span *AB*. Using the knowledge that the tangent at *A* is horizontal, i.e. $\theta_A = 0$, we look for the distance *x* from *A* that satisfies:

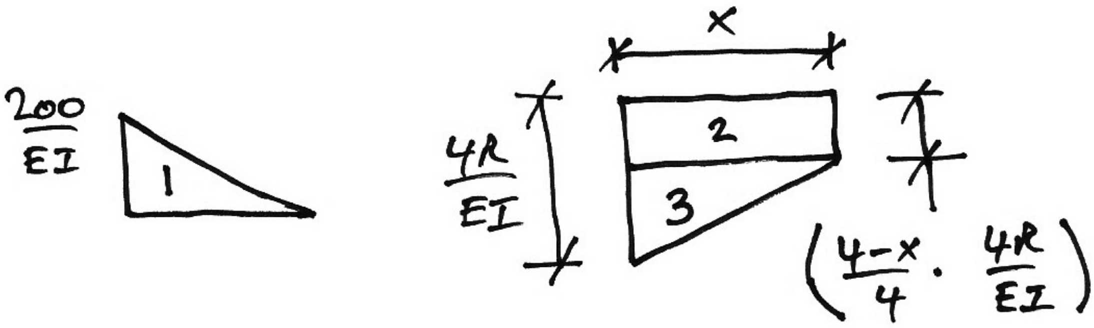
$$d\theta_{Ax} = \theta_A - \theta_x = 0$$

By inspection on the deflected shape, it is apparent that the maximum deflection occurs to the right of the point load. Hence we have the following:





So using Mohr I we calculate the change in rotation by finding the area of the curvature diagram between *A* and *x*. The diagram is split for ease:



The Area 1 is trivial:

$$A_1 = \frac{1}{2} \cdot 2 \cdot \frac{200}{EI} = \frac{200}{EI}$$

For Area 2, we need the height first which is:

$$h_2 = \frac{4-x}{4} \cdot \frac{4R}{EI} = \frac{4 \cdot 125 - 125}{4EI} = \frac{125}{EI} - \frac{125}{EI}x$$

And so the area itself is:

$$A_2 = x \cdot \left[\frac{125}{EI} - \frac{125}{EI}x \right]$$

For Area 3 the height is:

$$h_3 = \frac{125}{EI} - \left[\frac{125}{EI} - \frac{125}{EI}x \right] = \frac{125}{EI}x$$

And so the area is:

$$A_2 = \frac{1}{2} \cdot x \cdot \frac{125}{EI} x$$

Being careful of the signs for the curvatures, the total area is:

$$\begin{aligned} EId\theta_{Ax} &= -A_1 + A_2 + A_3 \\ &= -200 + x\left(125 - \frac{125}{4}x\right) + \frac{125}{8}x^2 \\ &= \left(\frac{125}{8} - \frac{125}{4}\right)x^2 + 125x - 200 \end{aligned}$$

Setting this equal to zero to find the location of the maximum deflection, we have:

$$\begin{aligned} -\frac{125}{8}x^2 + 125x - 200 &= 0 \\ 5x^2 - 40x + 64 &= 0 \end{aligned}$$

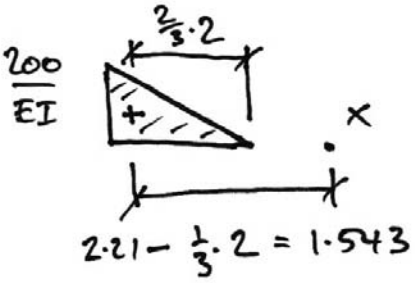
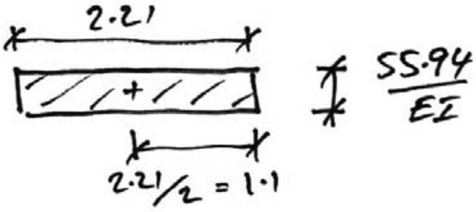
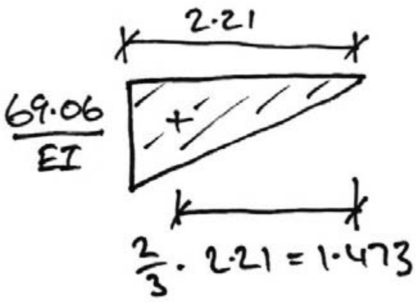
Thus, $x = 5.89$ m or $x = 2.21$ m. Since we are dealing with the portion *AB*, $x = 2.21$ m.

Find the maximum deflection

Since the tangent at both *A* and *x* are horizontal, i.e. $\theta_A = 0$ and $\theta_x = 0$, the deflection is given by:

$$\delta_{\max} = \Delta_{xA}$$

Using Mohr II and Areas 1, 2 and 3 as previous, we have:

Area 1		$A_1\bar{x}_1=-\frac{200}{EI}\cdot 1.543$ $=-\frac{308.67}{EI}$
Area 2		$h_2=\frac{4-2.21}{4}\cdot\frac{4R}{EI}=\frac{55.94}{EI}$ $A_2\bar{x}_2=2.21\cdot\frac{55.94}{EI}\cdot\frac{2.21}{2}$ $=\frac{136.61}{EI}$
Area 3		$h_3=2.21\cdot\frac{125}{EI}=\frac{69.06}{EI}$ $A_3\bar{x}_3=\left[\frac{1}{2}\cdot 2.21\cdot\frac{69.06}{EI}\right]\cdot 1.473$ $=\frac{112.43}{EI}$

Thus:

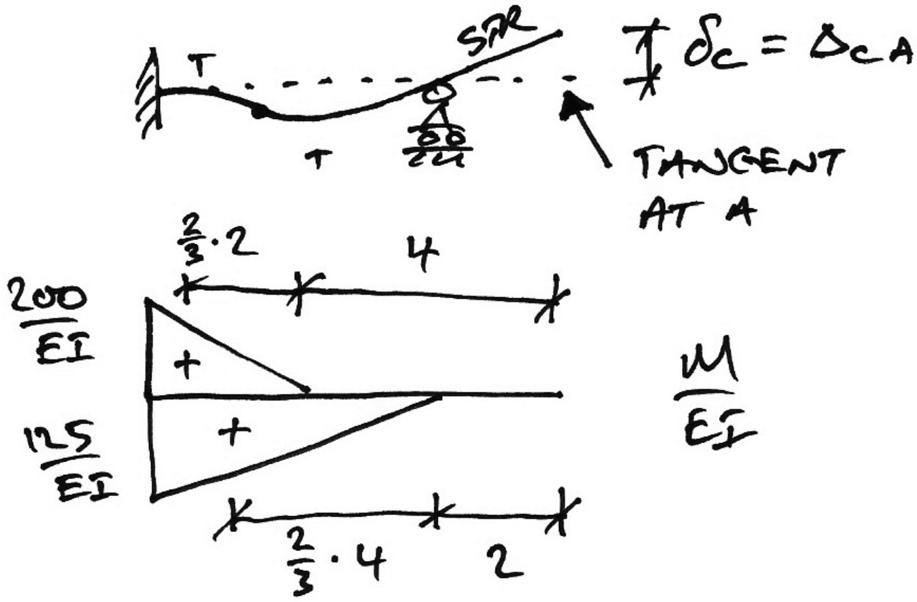
$$EI\Delta_{xB}=EI\delta_{\max}=-308.67+136.61+112.43$$
$$\Rightarrow \delta_{\max}=\frac{-59.63}{EI}$$

The negative sign indicates that the negative bending moment diagram dominates, i.e. the hogging of the cantilever is pushing the deflection downwards.

Find the deflection at C

For the deflection at *C* we again use the fact that $\theta_A = 0$ with Mohr II to give:

$$\delta_C = \Delta_{CA}$$



From the diagram we have:

$$EI\Delta_{CA} = -\left(\frac{1}{2}\cdot 2\cdot 200\right)\left(\frac{4}{3}+4\right)+\left(\frac{1}{2}\cdot 4\cdot 125\right)\left(2+\frac{8}{3}\right)$$

$$\delta_C = \frac{+100}{EI}$$

The positive sign indicates that the positive bending moment region dominates and so the deflection is upwards.

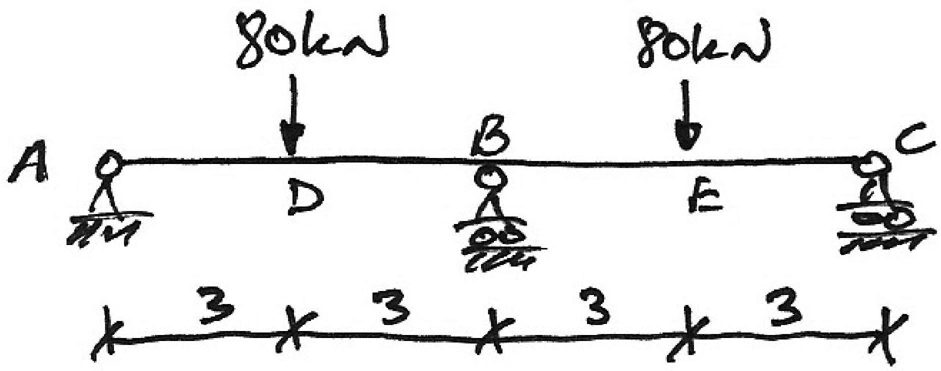
4.3 Example 7: 2-Span Beam

For the following beam of constant *EI*, using Mohr's theorems:

(a) Draw the bending moment diagram;

(b) Determine, δ_D and δ_E;

Give your answers in terms of *EI*.



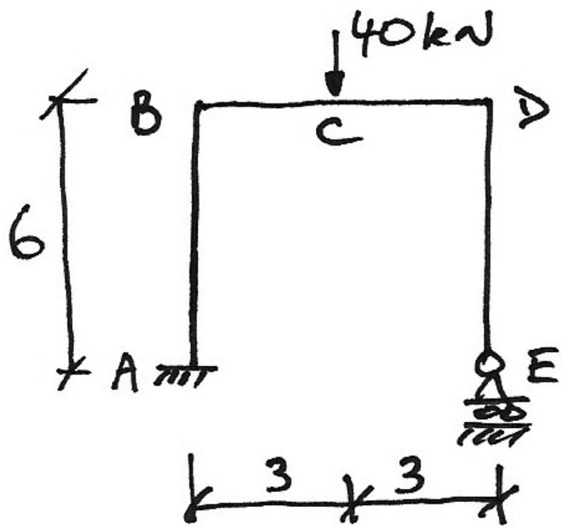
To be done in class.

4.4 Example 8: Simple Frame

For the following frame of constant $EI = 40 \text{ MNm}^2$, using Mohr's theorems:

(a) Draw the bending moment and shear force diagram;

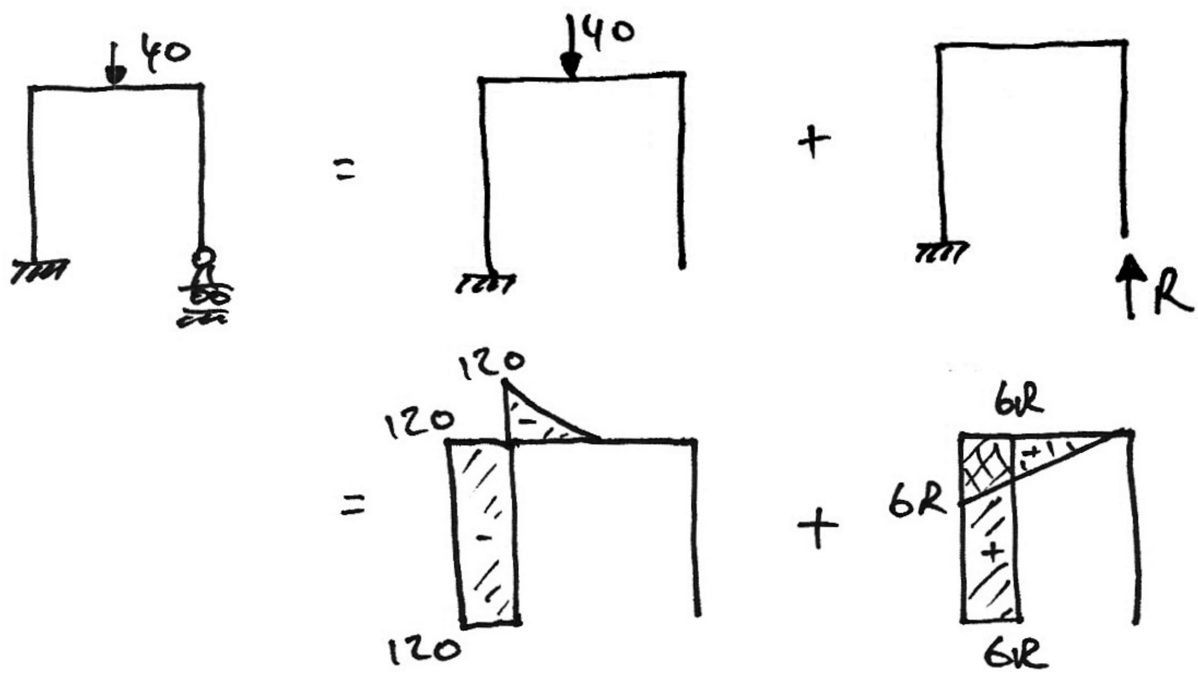
(b) Determine the horizontal deflection at *E*.



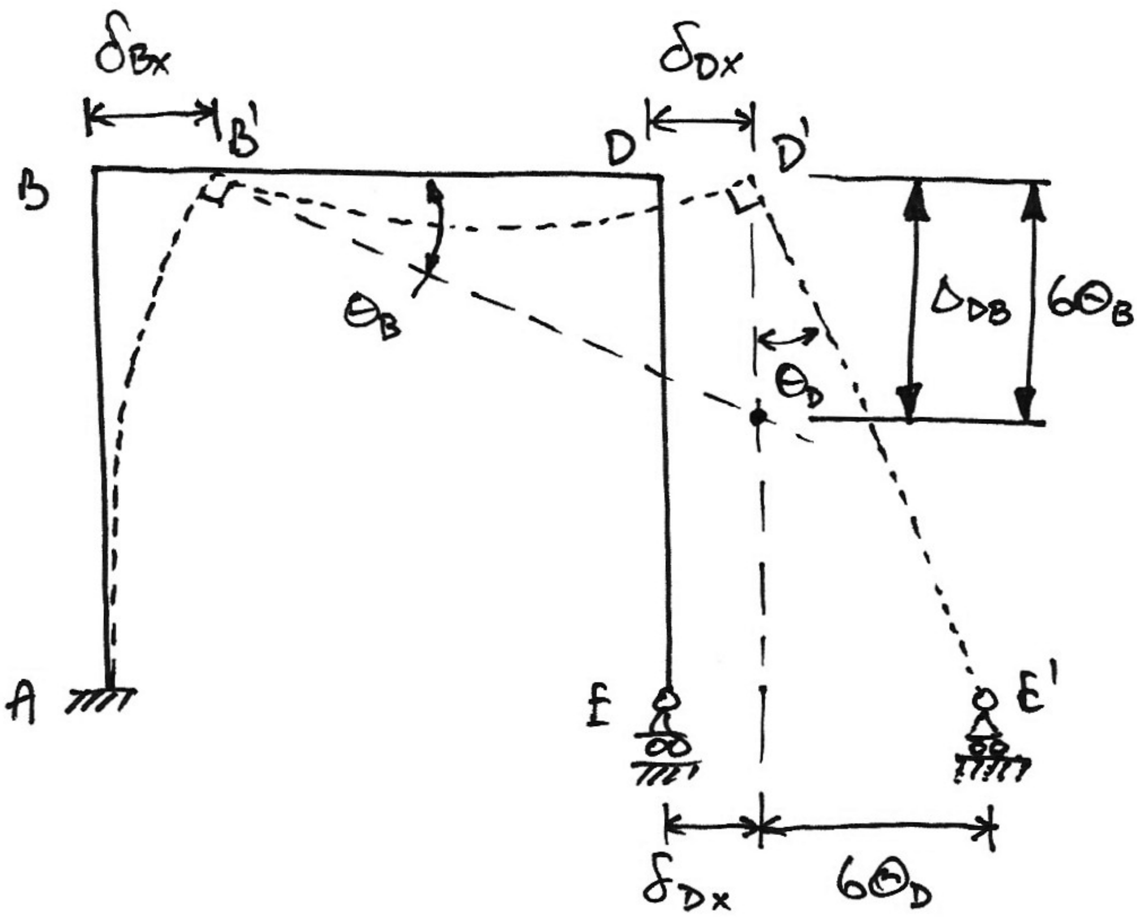
Part (a)

Solve for a Redundant

As with the beams, we split the structure into primary and reactant structures:



We also need to draw the deflected shape diagram of the original structure to identify displacements that we can use:

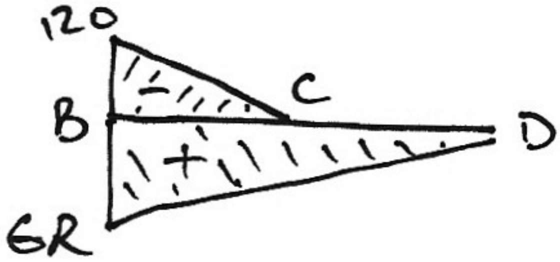


To solve for *R* we could use any known displacement. In this case we will use the vertical intercept Δ_{DB} as shown, because:

- We can determine Δ_{DB} for the original structure in terms of *R* using Mohr's Second Theorem;
- We see that $\Delta_{DB} = 6\theta_B$ and so using Mohr's First Theorem for the original structure we will find θ_B, again in terms of *R*;
- We equate the two methods of calculating Δ_{DB} (both are in terms of *R*) and solve for *R*.

Find Δ_{DB} by Mohr II

Looking at the combined bending moment diagram, we have:



$$\begin{aligned} EI\Delta_{DB} &= \left[\frac{1}{2}\cdot 6\cdot 6R\right]\cdot\left[\frac{2}{3}\cdot 6\right]-\left[\frac{1}{2}\cdot 3\cdot 120\right]\cdot\left[3+\frac{2}{3}\cdot 3\right] \\ &= 72R-900 \end{aligned}$$

Find θ_B by Mohr I

Since the tangent at *A* is vertical, the rotation at *B* will be the change in rotation from *A* to *B*:

$$\begin{aligned} d\theta_{BA} &= \theta_B - \theta_A \\ &= \theta_B - 0 \\ &= \theta_B \\ &= \text{Area of } \left(\frac{M}{EI}\right)_{B \text{ to } A} \end{aligned}$$

Therefore, by Mohr I:

$$\begin{aligned} EI\theta_B &= \text{Area of } \left(\frac{M}{EI}\right)_{B \text{ to } A} \\ &= 6\cdot 6R - 120\cdot 6 \\ &= 36R-720 \end{aligned}$$

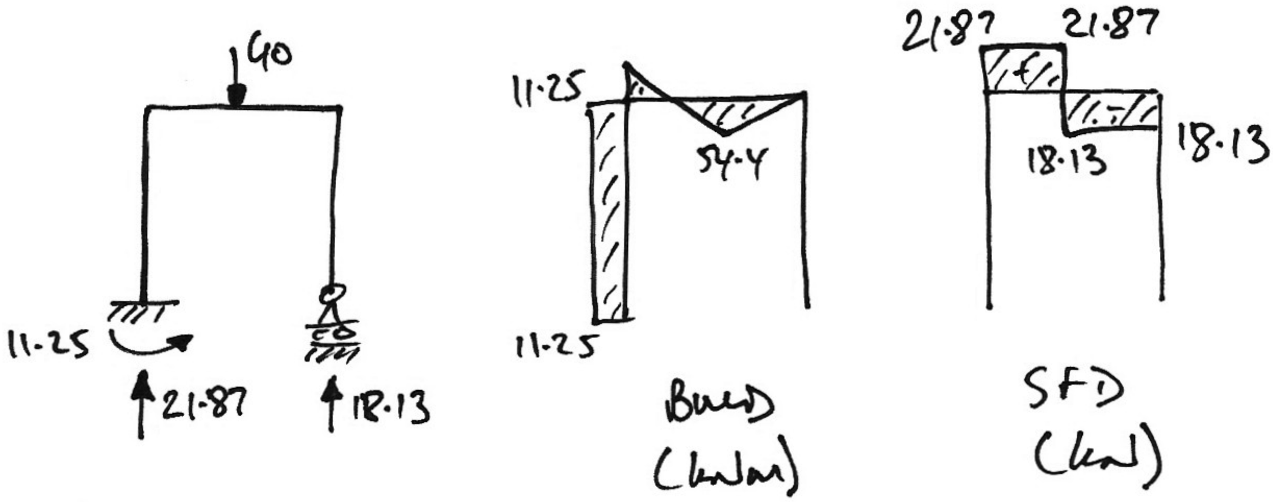
Equate and Solve for *R*

As identified previously:

$$\begin{aligned} \Delta_{DB} &= 6\theta_B \\ 72R - 900 &= 6\left[36R - 720\right] \\ R &= 18.13 \text{ kN} \end{aligned}$$

Diagrams

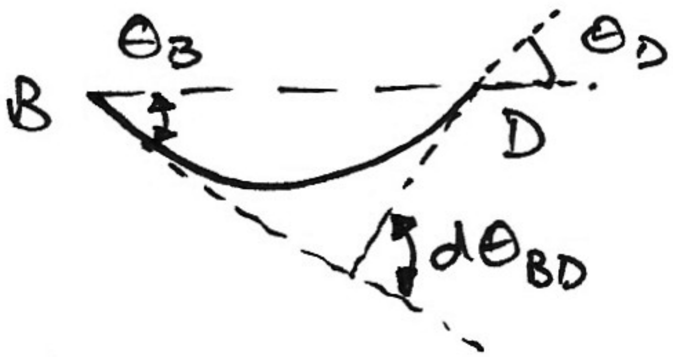
Knowing R we can then solve for the reactions, bending moment and shear force diagrams. The results are:



Part (b)

The movement at E is comprised of δ_{Dx} and $6\theta_D$ as shown in the deflection diagram. These are found as:

- Since the length of member BD doesn't change, $\delta_{Dx} = \delta_{Bx}$. Further, by Mohr II, $\delta_{Bx} = \Delta_{BA}$;
- By Mohr I, $\theta_D = \theta_B - d\theta_{BD}$, that is, the rotation at D is the rotation at B minus the change in rotation from B to D:



So we have:

$$\begin{aligned} EI\Delta_{BA} &= [6R \cdot 6][3] - [120 \cdot 6][3] \\ &= -202.5 \end{aligned}$$

$$\begin{aligned} EId\theta_{BD} &= \left[\frac{1}{2} \cdot 6R \cdot 6\right] - \left[\frac{1}{2} \cdot 120 \cdot 3\right] \\ &= 146.25 \end{aligned}$$

Notice that we still use the primary and reactant diagrams even though we know R. We do this because the shapes and distances are simpler to deal with.

From before we know:

$$EI\theta_B = 36R - 720 = 67.5$$

Thus, we have:

$$\begin{aligned} EI\theta_D &= EI\theta_B - d\theta_{BD} \\ &= 67.5 - 146.25 \\ &= -78.75 \end{aligned}$$

The minus indicates that it is a rotation in opposite direction to that of θ_B which is clear from the previous diagram. Since we have taken account of the sense of the rotation, we are only interested in its absolute value. A similar argument applies to the minus sign for the deflection at *B*. Therefore:

$$\begin{aligned} \delta_{Ex} &= \delta_{Bx} + 6\theta_D \\ &= \frac{202.5}{EI} + 6 \cdot \frac{78.75}{EI} \\ &= \frac{675}{EI} \end{aligned}$$

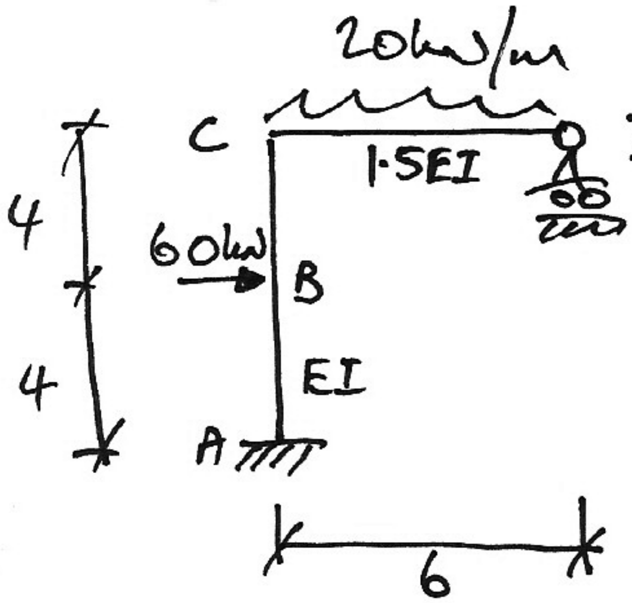
Using $EI = 40 \text{ MNm}^2$ gives $\delta_{Ex} = 16.9 \text{ mm}$.

4.5 Example 9: Complex Frame

For the following frame of constant $EI = 40\ \text{MNm}^2$, using Mohr's theorems:

(a) Draw the bending moment and shear force diagram;

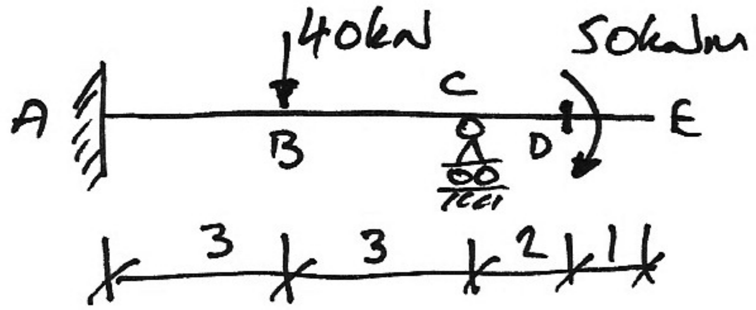
(b) Determine the horizontal deflection at *D*.



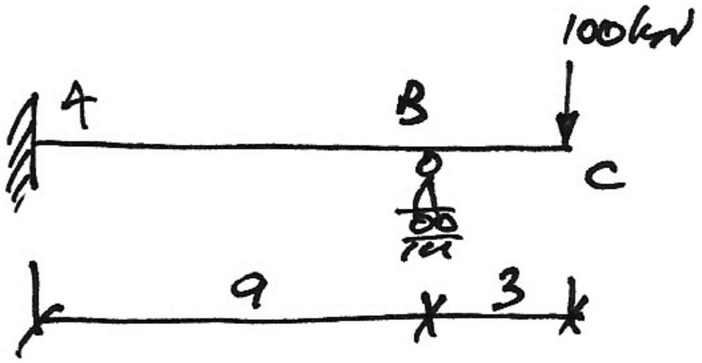
To be done in class.

4.6 Problems

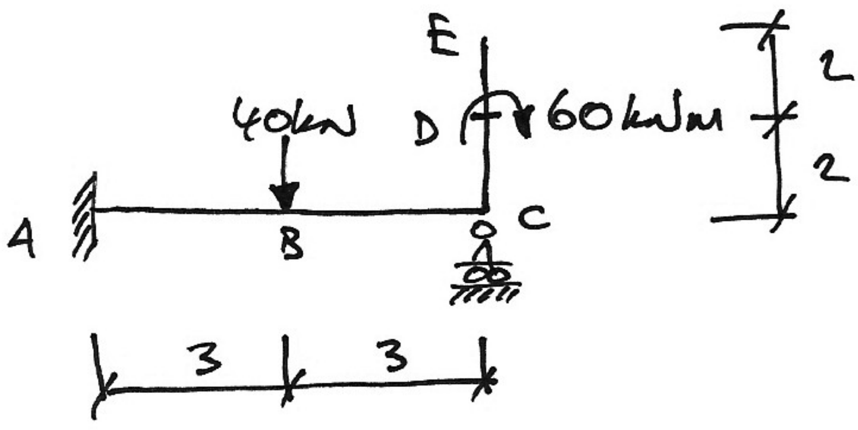
1. For the following prismatic beam, find the bending moment diagram and the rotation at *E* in terms of *EI*.



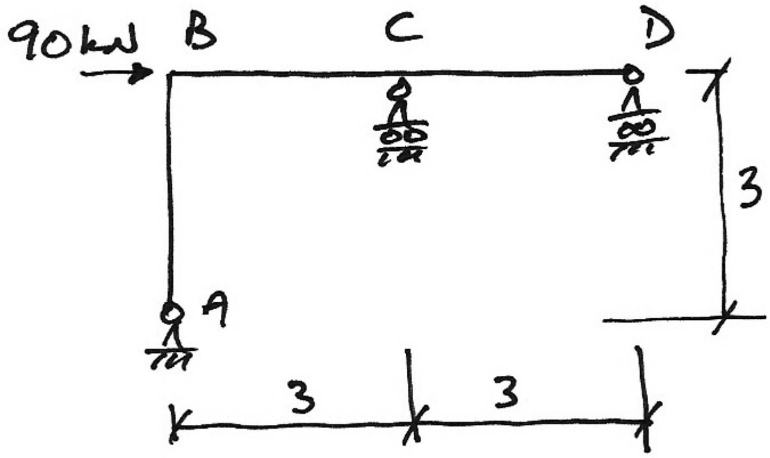
2. For the following prismatic beam, find the bending moment diagram and the rotation at *C* in terms of *EI*. *(Autumn 2007)*



3. For the following prismatic frame, find the bending moment and shear force diagrams and the horizontal deflection at *E* in terms of *EI*.



4. For the following prismatic frame, find the bending moment diagram and the horizontal deflection at *D* in terms of *EI*. *(Summer 2006)*



5. For the following prismatic frame, find the bending moment diagram and the horizontal defection at *C* in terms of *EI*. *(Summer 2007)*